W1. Michael Th. Rassias

Let $a, b \in \mathbb{N}$, with $b \geq 2$. If $a \neq 0 \pmod{b}$, then

$$\left\{\frac{a}{b}\right\} = \left\{\frac{a-1}{b}\right\} + \frac{1}{b},$$

If $a \equiv 0 \pmod{b}$, then

$$\left\{\frac{a-2}{b}\right\} = 1 - \frac{2}{b}.$$

Solution by Arkady Alt , San Jose, California, USA.

1. Let $a \neq 0 \pmod{b}$ then $\frac{a}{b} = k + \frac{r}{b}$ where $k = \left[\frac{a}{b}\right]$ and $r \in \{1, \ldots, b-1\}$.

Then $\left\{\frac{a}{b}\right\} = \frac{r}{b}$ and

$$\left\{\frac{a-1}{b}\right\} + \frac{1}{b} = \left\{k + \frac{r-1}{b}\right\} + \frac{1}{b} = \left\{\frac{r-1}{b}\right\} + \frac{1}{b} = \frac{r-1}{b} + \frac{1}{b} = \frac{r}{b}.$$

2. If $a \equiv 0 \pmod{b}$, then $\frac{a}{b} = k \Rightarrow \left\{\frac{a-2}{b}\right\} = \left\{k - \frac{2}{b}\right\} = \left\{-\frac{2}{b}\right\} = 1 - \frac{2}{b}$,

because in case $\{x\} \neq 0$ holds* $\{-x\} = 1 - \{x\}$.

* Since $\{x\} \neq 0$ then $x = [x] + \{x\}, 0 < \{x\} < 1, [x] \in \mathbb{Z}$ yield $-x = (-[x] - 1) + (1 - \{x\})$

Since any real x can be represented in the form $x = n + \tau$, where $n \in \mathbb{Z}$ and $\tau \in [0, 1)$ uniquely

and $x = [x] + \{x\}, 0 < \{x\} < 1, [x] \in \mathbb{Z}$ yield $-x = (-[x] - 1) + (1 - \{x\})$, where $-[x] - 1 \in \mathbb{Z}$,

$\{x\} \neq 0$ then $-[x] - 1 \in \mathbb{Z}, 1 - \{x\} \in (0, 1)$ and, therefore, $1 - \{x\} = \{-x\}$.